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Zeta-function regularization and the thermodynamic potential for quantum fields in symmetric spaces

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Abstract. We calculate a temperature-dependent part of the one-loop thermodynamic potential (and the free energy) for charged massive fields in a general class of irreducible rank 1 symmetric spaces. Both low- and high-temperature expansions are derived and the role of non-trivial topology influence on asymptotic properties of the potential is discussed.

1. Introduction

The problem of asymptotic expansions of the one-loop effective potential in Kaluza–Klein finite-temperature theories with non-vanishing chemical potential has been studied for a long time by several authors (for a review see [1–3]). The low- and high-temperature asymptotics in powers of $\beta = 1/T$, where T is the temperature of a system, has been evaluated in terms of integrated heat kernel coefficients, related to the scalar [4–7] and spinor [1–3] Laplacian acting on a smooth (compact) d-manifold M^d without boundary. An extension of this analysis to a Fermi gas was given first in [8]. The boundary conditions for a curved space in the thermodynamic system with non-vanishing chemical potential have been considered in [9, 10]. A finite-temperature analysis was also developed in manifolds with hyperbolic spatial sections [2, 3, 11–13]. The method of zeta-function regularization in the presence of the multiplicative anomaly for a system of charged bosonic fields with non-vanishing chemical potential was reconsidered and actively analysed in [14].

Only recently have the topological Casimir energy [15], the one-loop effective action, the multiplicative and the conformal anomaly [16, 17], associated with the product of Laplace-type operators acting in rank 1 symmetric spaces, been analysed. The goal of this paper is to study the influence of such non-trivial topology of manifolds M^d on the one-loop contribution to the thermodynamic potential and to the free energy of charged massive fields. We consider a general class of irreducible symmetric rank 1 Einstein d-manifolds. Geometric structure on Einstein manifolds M^d is related to an Einstein metric g, for which the equation $\mathrm{Ric}(g) = \Lambda g$ holds, where $\mathrm{Ric}(g)$ is the Ricci tensor and Λ is a constant.

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Trivial examples of Einstein manifolds are spaces with constant sectional curvature (in particular, uniform spaces $\mathbb{R}^d$, S^d and $\mathbb{H}^d$).

The contents of the paper are the following. In section 2 we review some relevant information on the spectral zeta function and the one-loop thermodynamic potential $\Omega^G(\beta, \mu)$ related to a non-compact simple split rank 1 Lie group G. The low-temperature expansion of the temperature-dependent part of the potential $\Omega_\beta^G(\beta, \mu)$ and the free energy are calculated in section 3. In section 4 the explicit form of the high-temperature expansion of $\Omega_\beta^G(\beta, \mu)$ is presented. We end with some conclusions in section 5. Finally, appendices A and B contain a summary of the heat kernel (appendix A) and the zeta function (appendix B) properties relevant to irreducible rank 1 symmetric spaces.

2. The one-loop thermodynamic potential

In this section general expressions for the thermodynamic potential and the free energy will be derived. For the sake of completeness we shall present the one-loop contributions of these thermodynamic quantities for the cases in which the spatial sections are general irreducible rank 1 symmetric spaces (Einstein manifolds) $X \equiv M^d$ of non-compact type. We recall that a Riemannian manifold (X, g) is a symmetric space if for any $x \in X$ there exist a group of manifold isometries Γ_x such that $\Gamma_x(x) = x$ and $T_x(\Gamma_x) = -Id_{(T_xX)}$, where T_xX is a tangent space. (An irreducible rank 1 Riemannian symmetric space (X, g) has the form G/K, where G is a rank 1 Lie group and $K \subset G$ is a maximal compact subgroup (see, for example, [18, 19]).)

We start with the thermodynamic potential for massive charged scalar fields with a non-vanishing chemical potential μ in thermal equilibrium at finite temperature in an ultra-static spacetime with spatial sector of the form $X = G/K$. Let $\Gamma \subset G$ be a discrete, co-compact, torsion-free subgroup. Let χ be a finite-dimensional unitary representation of Γ, let $\{\lambda_l\}_{l=0}^{\infty}$ be the set of eigenvalues of the second-order operator of Laplace type $\mathcal{L}_0 = -\Delta_\Gamma$ acting on smooth sections of the vector bundle over $\Gamma\backslash X$ induced by χ, and let $n_l(\chi)$ denote the multiplicity of λ_l.

For the ultra-static spacetime with topology $S^1 \bigotimes X$, the elliptic second-order differential operator $L(\mu)$ is a matrix-valued operator acting on the real and imaginary part of the complex scalar charged field. It has the form $L(\mu) = \text{diag}\left(-(\partial_\tau - e\mu)^2 + \mathcal{L}, -(\partial_\tau + e\mu)^2 + \mathcal{L}\right) \equiv \text{diag}(\mathcal{O}_+, \mathcal{O}_-)$; to simplify the calculation we take $e = 1$, where e is an elementary charge. The operators $\mathcal{O}_\pm$ are not Hermitian; in fact, they are normal and their eigenvalues are complex and read

$$\left(\frac{2\pi n}{\beta} \pm \mathrm{i}\mu\right)^2 + \lambda_l + b \qquad n \in \mathbb{Z}. \tag{2.1}$$

In equation (2.1) b is an arbitrary constant (an endomorphism of the vector bundle over $\Gamma\backslash X$). We shall also need a suitable regularization of the determinant of an elliptic differential operator, and we shall make a choice of the zeta-function regularization. The zeta function associated with the operator $\mathcal{L} \equiv \mathcal{L}_0 + b$ has the form

$$\zeta_\Gamma(s|\mathcal{L}) = \sum_l n_l(\chi)\{\lambda_l + b\}^{-s} \tag{2.2}$$

where $\zeta_\Gamma(s|\mathcal{L})$ is a well defined analytic function for $\text{Re}\, s > \dim(X)/2$, and it can be analytically continued to a meromorphic function on the complex plane $\mathbb{C}$, regular at $s = 0$.

The canonical partition function can be written as follows:

$$\log Z(\beta, \mu) = -\beta\Omega^G(\beta, \mu) = -S_c[\phi_c, g] - \tfrac{1}{2}\log\det[L(\mu)] \tag{2.3}$$

where ϕ_c is a solution, which extremizes the classical action $S_c[\phi_c, g]$; equation (2.3) defines the thermodynamic potential $\Omega^G(\beta, \mu)$. If one makes use of zeta-function regularization, one obtains $\zeta'_\Gamma(0|\mathcal{O}_-) = \zeta'_\Gamma(0|\mathcal{O}_+)$ and

$$\Omega^G(\beta, \mu) = \frac{1}{\beta} S_c[\phi_c, g] - \frac{1}{\beta}\zeta'_\Gamma(0|\mathcal{O}_+) - \frac{1}{2\beta}\mathcal{A}(\mathcal{O}_+, \mathcal{O}_-) \tag{2.4}$$

where $\mathcal{A}(\mathcal{O}_+, \mathcal{O}_-)$ is the related multiplicative anomaly [14]. The explicit form of the $\mathcal{A}(\mathcal{L}_0 + b_1, \mathcal{L}_0 + b_2)$ associated with spaces X listed in equation (A.3) of appendix A, can be found in [16].

Using the Mellin representation for the zeta function one can obtain useful formulae for the non-trivial temperature-dependent part $\Omega^G_\beta(\beta, \mu)$ of the thermodynamic potential (for details see [3, 14])

$$\begin{aligned}
\Omega^G_\beta(\beta, \mu) &\equiv \Omega^G(\beta, \mu) - \Omega^G_0 - \frac{1}{2\beta}\mathcal{A}(\mathcal{O}_+, \mathcal{O}_-) \\
&= -\frac{1}{\pi}\sum_{\nu=1}^{\infty}\int_{\mathbb{R}} e^{i\nu\beta t}\zeta'_\Gamma\big(0\big|\mathcal{L} + [t + i\mu]^2\big)\,dt \\
&= -\frac{1}{\sqrt{\pi}}\sum_{\nu=1}^{\infty}\cosh(\nu\beta\mu)\int_0^\infty t^{-3/2}e^{-\nu^2\beta^2/4t}\omega_\Gamma(t; b, \chi)\,dt \qquad (2.5)\\
&= -\frac{1}{\pi i}\sum_{\nu=0}^{\infty}\frac{\mu^{2\nu}}{(2\nu)!}\int_{\mathrm{Re}\, s=c}\zeta_R(s)\,\Gamma(s + 2\nu - 1)\,\zeta_\Gamma\left(\frac{s + 2\nu - 1}{2}\middle|\mathcal{L}\right)\beta^{-s}\,ds \qquad (2.6)
\end{aligned}$$

where

$$\Omega^G_0 = \frac{1}{\beta}S_c[\phi_c, g] + \xi^{(r)}\big(-\tfrac{1}{2}\big|\mathcal{L}\big) \tag{2.7}$$

$$\xi^{(r)}\big(-\tfrac{1}{2}\big|\mathcal{L}\big) = \mathrm{PP}\,\zeta_\Gamma\big(-\tfrac{1}{2}\big|\mathcal{L}\big) + (2 - 2\log 2)\,\mathrm{Res}\,\zeta_\Gamma\big(-\tfrac{1}{2}\big|\mathcal{L}\big). \tag{2.8}$$

In equation (2.5) $\omega_\Gamma(t; b, \chi)$ is the heat kernel of an operator $\mathcal{L}$ (see equations (A.1), (A.2), (A.6) and (A.8) of appendix A); the symbols PP and Res in equation (2.8) stand for the finite part and the residue of the function at the special point, respectively. The formulae (2.5) and (2.6) are valid for a charged scalar field. In the case of a neutral scalar field we have to multiply all results by a factor of $\frac{1}{2}$.

Different representations of the temperature-dependent part of the thermodynamic potential can be obtained by means of equations (2.5) and (2.6). In fact, using equation (A.6) and (A.7) for the heat kernel in (2.5) we obtain

$$\begin{aligned}
\Omega^G_\beta(\beta, \mu) = -\frac{1}{\sqrt{\pi}}\sum_{\nu=1}^{\infty}\cosh(\nu\beta\mu)\int_0^\infty t^{-3/2}e^{-\nu^2\beta^2/4t} \\
\times\left[V\int_{\mathbb{R}} e^{-(r^2+\alpha^2)t}|C(r)|^{-2}\,dr + \theta_\Gamma(t; b, \chi)\right]dt. \qquad (2.9)
\end{aligned}$$

For the sake of generality we shall set $b + \rho_0^2 \equiv \alpha^2$, where α is an arbitrary constant. Note that $\alpha^2 = \rho_0^2$ and $\alpha^2 = 0$ correspond to the massless and conformal coupling case, respectively. Taking into account an integral representation for the MacDonald functions $K_\nu(z)$:

$$K_\nu(z) = \tfrac{1}{2}\big(\tfrac{1}{2}z\big)^\nu\int_0^\infty e^{-t-z^2/4t}t^{-\nu-1}\,dt \qquad \big(|\arg z| < \pi/2 \quad \text{and} \quad \mathrm{Re}\, z^2 > 0\big) \tag{2.10}$$

we get

$$\Omega_\beta^G(\beta,\mu) = -2\sqrt{\frac{2}{\pi}}\sum_{\nu=1}^{\infty}\cosh(\nu\beta\mu)\Bigg[V\int_{\mathbb{R}}|C(r)|^{-2}\left(\frac{\sqrt{r^2+\alpha^2}}{\nu\beta}\right)^{1/2}K_{1/2}\Big(\nu\beta\sqrt{r^2+\alpha^2}\Big)\,\mathrm{d}r + \frac{1}{\sqrt{2\pi}}\sum_{\gamma\in C_\Gamma-\{1\}}\chi(\gamma)\,t_\gamma\,j(\gamma)^{-1}\,C(\gamma)\frac{\alpha}{\sqrt{\nu^2\beta^2+t_\gamma^2}}K_1\Big(\alpha\sqrt{\nu^2\beta^2+t_\gamma^2}\Big)\Bigg]. \tag{2.11}$$

3. The low-temperature expansion

3.1. The thermodynamic potential

Equations (2.5) and (2.6) are useful for the low- and high-temperature expansion of the thermodynamic quantity [3]. Indeed, in order to specialize equation (2.5) for the low-temperature case let us recall the asymptotic of the MacDonald functions for real values z and ν, namely

$$K_\nu(z\mapsto\infty)\simeq\sqrt{\frac{\pi}{2z}}\mathrm{e}^{-z}\sum_{k=0}^{\infty}\frac{\Gamma\big(\nu+k+\frac{1}{2}\big)}{\Gamma(k+1)\,\Gamma\big(\nu-k+\frac{1}{2}\big)}(2z)^{-k}. \tag{3.1}$$

As a result we have

$$\Omega_\beta^G(\beta\mapsto\infty,\mu)\simeq-\sum_{\nu=1}^{\infty}\Bigg[\frac{V}{\nu\beta}\int_{\mathbb{R}}|C(r)|^{-2}\mathrm{e}^{-\nu\beta(\sqrt{r^2+\alpha^2}-|\mu|)}\,\mathrm{d}r + \sqrt{\frac{\alpha}{2\pi}}\sum_{\gamma\in C_\Gamma-\{1\}}\sum_{k=0}^{\infty}\frac{\chi(\gamma)\,t_\gamma\,j(\gamma)^{-1}\,C(\gamma)}{(2\alpha)^k\big(\nu^2\beta^2+t_\gamma^2\big)^{k/2+3/4}}\frac{\Gamma\big(k+\frac{3}{2}\big)}{\Gamma(k+1)\,\Gamma\big(\frac{3}{2}-k\big)} \times\mathrm{e}^{-\alpha\sqrt{\nu^2\beta^2+t_\gamma^2}+\nu\beta|\mu|}\Bigg]. \tag{3.2}$$

Finally, using the explicit form of the Harish–Chandra–Plancherel measure (B.5) in equation (3.2) after straightforward calculation we get

$$\Omega_\beta^G(\beta\mapsto\infty,\mu)\simeq\frac{2\pi C_G V}{\beta}\sum_{\nu=1}^{\infty}\sum_{l=0}^{d/2-1}\frac{a_{2l}}{\nu}\int_{\mathbb{R}}r^{2l+1}\mathrm{e}^{-\nu\beta(\sqrt{r^2+\alpha^2}-|\mu|)}\mathcal{R}_G(r)\,\mathrm{d}r - \sqrt{\frac{\alpha}{2\pi}}\sum_{\nu=1}^{\infty}\sum_{\gamma\in C_\Gamma-\{1\}}\sum_{k=0}^{\infty}\frac{\chi(\gamma)\,t_\gamma\,j(\gamma)^{-1}\,C(\gamma)}{(2\alpha)^k\big(\nu^2\beta^2+t_\gamma^2\big)^{k/2+3/4}} \times\frac{\Gamma\big(k+\frac{3}{2}\big)}{\Gamma(k+1)\,\Gamma\big(\frac{3}{2}-k\big)}\mathrm{e}^{-\alpha\sqrt{\nu^2\beta^2+t_\gamma^2}+\nu\beta|\mu|} \tag{3.3}$$

where

$$\mathcal{R}_G(r)=\begin{bmatrix}\big(1+\mathrm{e}^{2\pi r}\big)^{-1}, & G=SO_1(2n,1),\\ \big(1+\mathrm{e}^{\pi r}\big)^{-1}, & G=SP(m,1),\,F_{4(-20)},\,SU(m,1)\text{ for odd }m,\\ \big(1-\mathrm{e}^{\pi r}\big)^{-1}, & G=SU(m,1)\text{ for even }m,\\ -(2r)^{-1}, & G=SO_1(2n+1,1).\end{bmatrix} \tag{3.4}$$

The leading terms come from the 'topological' part of the thermodynamic potential, related to the function $\theta_\Gamma(t;b,\chi)$, which is determined by equation (A.8).

3.2. The free energy

The one-loop free energy can be derived from the thermodynamic potential (3.3) in the limit $\mu \mapsto 0$. Thus the formulae for the free energy can be considered as a particular case of expressions given for thermodynamic potentials. The low-temperature contribution to the one-loop free energy has the form

$$\Omega_\beta^G(\beta \mapsto \infty, 0) \simeq \frac{2A}{\beta} \sum_{l=0}^{d/2-1} a_{2l} \int_{\mathbb{R}} r^{2l+1} \left[\beta\sqrt{r^2+\alpha^2} - \log\left(e^{\beta\sqrt{r^2+\alpha^2}} - 1\right)\right] \mathcal{R}_G(r)\, dr$$
$$- \frac{1}{\sqrt{2\pi}} \sum_{\nu=1}^{\infty} \sum_{k=0}^{\infty} \sum_{\gamma \in C_\Gamma - \{1\}} \frac{\chi(\gamma)\, t_\gamma\, j(\gamma)^{-1}\, C(\gamma)}{(2\alpha)^{k-\frac{1}{2}} \left(\nu^2\beta^2 + t_\gamma^2\right)^{k/2+3/4}}$$
$$\times \frac{\Gamma\left(k+\frac{3}{2}\right)}{\Gamma(k+1)\,\Gamma\left(\frac{3}{2}-k\right)} e^{-\alpha\sqrt{\nu^2\beta^2+t_\gamma^2}} \tag{3.5}$$

where $A = \pi C_G V$.

4. The high-temperature expansion

For the high-temperature expansion it is convenient to use the Mellin–Barnes representation (2.6) and integrate it on a closed path enclosing a suitable number of poles. To carry out the integration first of all we shall consider the simplest case of $G = SO_1(2n+1, 1)$ in (B.5).

4.1. The group $G = SO_1(2n+1, 1)$

Taking into account equation (2.6) we recall that the zeta function in a $(2n+1)$-dimensional smooth manifold without boundary has simple poles at the points $s = (2n+1)/2 - k$, $k \in \mathbb{N}$, [20]. The Riemann zeta function $\zeta_R(s)$ has a simple pole at $s = 1$ and simple zeros at all the negative even numbers while the function $\Gamma(s)$ has simple poles at $s = -k$, $k \in \mathbb{N}$. The temperature dependent part of the thermodynamic potential can be written as follows:

$$\Omega_\beta^{SO_1(2n+1,1)}(\beta \mapsto 0, \mu) = -\frac{1}{\pi i} \sum_{\nu=0}^{\infty} \frac{\mu^{2\nu}}{(2\nu)!} \int_{\mathrm{Re}\, s=c} \mathcal{F}^{SO_1(2n+1,1)}(s; \nu, \beta)\, ds \tag{4.1}$$

where

$$\mathcal{F}^{SO_1(2n+1,1)}(s; \nu, \beta) = 2^{s+2\nu-2} \pi^{-1/2} \Gamma\left(\tfrac{1}{2}s + \nu\right) \zeta_R(s)\, \beta^{-s}$$
$$\times \left[A \sum_{j=0}^{n} a_{2j} \alpha^{2j-2\nu-s+2} \Gamma\left(j+\tfrac{1}{2}\right) \Gamma\left(\tfrac{1}{2}s + \nu - j - 1\right)\right.$$
$$\left. + T_\Gamma\left(\frac{s+2\nu-1}{2}; \alpha, \chi\right)\right]. \tag{4.2}$$

The meromorphic integrand $\mathcal{F}^{SO_1(2n+1,1)}(s; \nu, \beta)$ has simple poles at the points $s = 1$ $(\nu \in \mathbb{N})$, $s = -2(\nu + k)$ $(k \in \mathbb{N}, k = -1, \ldots, -n-1)$ and $s = 0$ $(\nu = 1, 2, \ldots, n+1)$. Moreover, for $\nu = 0$ at $s = 0$ we have simple and double poles. Thus choosing a contour of integration in the left half-plane we obtain the high-temperature expansion in the form

$$
\begin{aligned}
\Omega_\beta^{SO_1(2n+1,1)}(\beta \mapsto 0, \mu) = -\frac{A}{\sqrt{\pi}}\Bigg\{ &\sum_{k=1}^{n+1}\sum_{\nu=0}^{k-1}\sum_{j\geqslant k-1}^{n}(-1)^{j+1-k}2^{2k-1}a_{2j} \\
&\times \frac{\mu^{2\nu}\Gamma(k)\Gamma\big(j+\frac{1}{2}\big)}{(2\nu)!\,\Gamma(j+2-k)}\zeta_R(2k-2\nu)\alpha^{2j-2k+2}\beta^{2(\nu-k)} \\
&+\Bigg[\sum_{\nu=0}^{\infty}\sum_{j=0}^{n}2^{2\nu-1}a_{2j}\frac{\mu^{2\nu}\Gamma\big(\nu+\frac{1}{2}\big)}{(2\nu)!}\Gamma\big(j+\tfrac{1}{2}\big)\Gamma\big(\nu-j-\tfrac{1}{2}\big)\alpha^{2j-2\nu+1} \\
&+\int_0^\infty \Psi_\Gamma(t+\rho_0+\alpha;\chi)\,\mathrm{d}t\Bigg]\beta^{-1} \\
&+\sum_{j=0}^{n}(-1)^{j+1}a_{2j}\frac{\Gamma\big(j+\frac{1}{2}\big)}{\Gamma(j+2)}\alpha^{2j+2}\big[\gamma_E+\log\big(\tfrac{1}{2}\alpha^2\beta\big)\big] \\
&+\sum_{\nu=1}^{n+1}\sum_{j\geqslant\nu-1}^{n}(-1)^{j-\nu}2^{2\nu-2}a_{2j}\frac{\mu^{2\nu}\,\Gamma(\nu)\,\Gamma\big(j+\frac{1}{2}\big)}{(2\nu)!\,\Gamma(j+2-\nu)}\alpha^{2j+2}\Bigg\} \\
&+\frac{1}{2\pi}\int_0^\infty \Psi_\Gamma(t+\rho_0+\alpha;\chi)\big(2\alpha t+t^2\big)^{1/2}\,\mathrm{d}t+\mathcal{O}\big(\beta^2\big)
\end{aligned}
\qquad (4.3)
$$

where γ_E is the Euler constant,

$$
\begin{aligned}
\mathcal{O}\big(\beta^2\big) = -\frac{A}{\sqrt{\pi}}&\sum_{\substack{\nu,k=0,\\ n+k\neq 0}}^{\infty}\sum_{j=0}^{n}(-1)^{k+j+1}a_{2j}\frac{\mu^{2\nu}\,\Gamma\big(j+\frac{1}{2}\big)}{2^{2k+1}\,(2\nu)!\,\Gamma(j+k+2)} \\
&\times\Gamma(-k)\,\zeta_R(-2k-2\nu)\,\alpha^{2j+2k+2}\beta^{2(\nu+k)}.
\end{aligned}
\qquad (4.4)
$$

At high temperature and even for zero chemical potential 'topological' terms give a contribution to the potential. The high-temperature expansion (4.3) in principal looks quite similar to the one obtained in [3] for $X = \mathbb{H}^3/\Gamma$.

For the case of minimally coupled scalar field in manifolds $S^1 \otimes M^d$ we have $\alpha^2 = m^2 + \rho_0^2$, where $b = m^2$, m is a mass of field. For example, when $n = 1$, ($d = 3$), the leading term of the Laurent series (4.3) has the form $-4Aa_{21}\pi^4/\big(90\beta^4\big)$, which is a known result [7, 9].

4.2. The group $G \neq SO_1(2n+1, 1), SU(d/2, 1)$

For $G = SO_1(2n, 1), SU(2p+1), SP(m, 1), F_{4(-20)}$ the integrand in equation (2.6) has the form

$$
\begin{aligned}
\mathcal{F}^G(s;\nu,\beta) = \zeta_R(s)\beta^{-s}\Bigg[&\frac{Aa(G)\,\Gamma(s+2\nu-1)}{2}W\bigg(\frac{s+2\nu-1}{2};\alpha^2,a(G)\bigg) \\
&+\frac{2^{s+2\nu-2}}{\sqrt{\pi}}\Gamma\Big(\frac{s}{2}+\nu\Big)T_\Gamma\bigg(\frac{s+2\nu-1}{2};\alpha,\chi\bigg)\Bigg]
\end{aligned}
\qquad (4.5)
$$

where $W(s;\alpha,a(G))$ is given by equation (B.7). Therefore the temperature-dependent part of the thermodynamic potential is

$$\Omega_\beta^G(\beta\mapsto 0,\mu) = -a(G)A\sum_{\nu=0}^{\infty}\sum_{m,j=0}^{d/2-1} a_j\frac{\mu^{2\nu}}{(2\nu)!}j!\,\Gamma(2m+2)\,\zeta_R(2m+3-2\nu)$$

$$\times\sum_{l=m}^{j}\frac{\mathcal{K}_{j-l}\big(m-l;\alpha^2,a(G)\big)}{(j-l)!}\prod_{\substack{q=0,\\ q\neq m}}^{l}(m-q)^{-1}\beta^{2\nu-2m-3}$$

$$-a(G)A\big[-W\big(0;\alpha^2,a(G)\big)\log\beta + W'\big(0;\alpha^2,a(G)\big)\big]\beta^{-1}$$

$$+a(G)A\sum_{\nu=1}^{d/2}\frac{\mu^{2\nu}\,\Gamma(2\nu)}{(2\nu)!}\big[U\big(\nu;\alpha^2,a(G)\big)+\gamma_E V\big(\nu;\alpha^2,a(G)\big)$$

$$-V\big(\nu;\alpha^2,a(G)\big)\log\beta + \psi'(2\nu)V\big(\nu;\alpha^2,a(G)\big)\big]\beta^{-1}$$

$$+\beta^{-1}\int_0^\infty \Psi_\Gamma(t+\rho_0+\alpha;\chi)\,\mathrm{d}t$$

$$+\frac{1}{2\pi}\int_0^\infty \Psi_\Gamma(t+\rho_0+\alpha;\chi)\big(2\alpha t+t^2\big)^{1/2}\,\mathrm{d}t + \mathcal{O}(\beta) \qquad (4.6)$$

where $\psi(s)\equiv\Gamma'(s)/\Gamma(s)$,

$$\mathcal{O}(\beta) = -a(G)A\sum_{\nu=0}^{\infty}\sum_{k=1}^{\infty}\frac{\mu^{2\nu}}{(2\nu)!(2k)!}\zeta_R(1-2\nu-2k)W\big(-k;\alpha^2,a(G)\big)\beta^{2\nu+2k-1} \qquad (4.7)$$

$$U\big(s;\alpha^2,a(G)\big) = \sum_{j=0}^{d/2-1}\sum_{\substack{l=0,\\ l<n-1}}^{j} a_{2j}\,j!\frac{\mathcal{K}_{j-l}\big(s-l-1;\alpha^2,a(G)\big)}{(j-l)!(s-1)(s-2)\ldots(s-(l+1))}$$

$$V\big(s;\alpha^2,a(G)\big) = \sum_{j=0}^{d/2-1}\sum_{l\geqslant n-1}^{j} a_{2j}\,j!\frac{\mathcal{K}_{j-l}\big(s-l-1;\alpha^2,a(G)\big)}{(j-l)!(s-1)(s-2)\ldots(s-(l+1))} \qquad (4.8)$$

and

$$W'\big(0;\alpha^2,a(G)\big) = \sum_{j=0}^{d/2-1}\sum_{l=0}^{j}\frac{a_{2j}\,j!(-1)^{l+1}}{(j-l)!(l+1)!}\bigg[\mathcal{K}'_{j-l}\big(-l-1;\alpha^2,a(G)\big)$$

$$+\tfrac{1}{2}\mathcal{K}_{j-l}\big(-l-1;\alpha^2,a(G)\big)\sum_{m=1}^{l+1}\frac{1}{m}\bigg]. \qquad (4.9)$$

4.3. The Group $G = SU(p,1)$

Finally, for $G=SU(p,1)$, $d=2p$, one gets

$$\Omega_\beta^{SU(p,1)}(\beta\mapsto 0,\mu) = \Omega_\beta^G(\beta\mapsto 0,\mu)$$

$$-2A\sum_{j=0}^{p-1}a_{2j}\bigg[\sum_{\nu=1}^{\infty}\frac{\mu^{2\nu}\Gamma(2\nu)}{(2\nu)!}\mathcal{J}_j\big(\nu;\alpha^2,\tfrac{1}{2}\pi\big)+\mathcal{J}'_j\big(0;\alpha^2,\tfrac{1}{2}\pi\big)\bigg]\beta^{-1}$$

$$-2A\sum_{\nu=1}^{\infty}\sum_{k=0}^{\infty}\sum_{j=0}^{p-1}a_{2j}\frac{\mu^{2\nu}\zeta_R(1-2\nu-2k)}{(2\nu)!(2k)!}\mathcal{J}_j\big(-k;\alpha^2,\tfrac{1}{2}\pi\big)\beta^{2\nu+2k-1}. \qquad (4.10)$$

In equation (4.10) $G\neq SO_1(2n+1,1)$, $SU(d/2,1)$ and $a(G)=\pi/2$ has been chosen.

5. Conclusions

In this paper an extension of previous results to the case in which the chemical potential for quantum fields in irreducible symmetric spaces of rank 1 is present has been proposed. In the case of low and high temperature we obtain a generalization of the results discussed in [1–3].

For the vector (spin 1) field the Hodge–de Rham operator $(d\delta + \delta d)$ acting on the exact one-forms is associated with the massless operator $[-\nabla^\mu\nabla^\nu + (d-1)]g_{\mu\nu}$. The eigenvalues of the operator are $\lambda_l^2 + (\rho_0 - 1)^2$ and for the Proca field of mass m we find $\alpha^2 = m^2 + (\rho_0 - 1)^2$.

Our results can also be extended to spin-$\frac{1}{2}$ (fermion) field, for which spin structure on a manifold has to be taken into account. Note that different spin structures are parametrized by the first cohomology group $H^1(X;\mathbb{Z}_2)$. Asymptotic expansions for the spin-$\frac{1}{2}$ field can be obtained using the relation $\Omega_F(\beta,\mu) = 2\Omega_B(2\beta,\mu) - \Omega_B(\beta,\mu)$ [3, 9], where the symbol F (B) stands for the fermion (boson) degree of freedom.

We hope that the proposed analysis of the one-loop thermodynamic properties of the potential will be interesting in view of future applications to concrete problems in quantum field theory at finite temperature, in quantum gravity (see [21]), in multidimensional cosmological models and in mathematical applications as well.

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Appendix A. The heat kernel

One can define the heat kernel of the elliptic operator $\mathcal{L}$ by

$$\omega_\Gamma(t;b,\chi) \equiv \mathrm{Tr}\left(\mathrm{e}^{-t\mathcal{L}}\right) = \frac{-1}{2\pi\mathrm{i}}\,\mathrm{Tr}\int_{\mathcal{C}_0} \mathrm{d}z\,\mathrm{e}^{-zt}(z-\mathcal{L})^{-1} \tag{A.1}$$

where $\mathcal{C}_0$ is an arc in the complex plane $\mathbb{C}$. By standard results in operator theory there exist $\epsilon, \delta > 0$ such that for $0 < t < \delta$ the heat kernel expansion holds

$$\omega_\Gamma(t;b,\chi) = \sum_{l=0}^{\infty} n_l(\chi)\,\mathrm{e}^{-(\lambda_l+b)t} = \sum_{0\leqslant l\leqslant l_0} a_l(\mathcal{L})t^{-l} + \mathrm{O}\left(t^\epsilon\right). \tag{A.2}$$

The following representations of X up to local isomorphism can be chosen

$$X = \begin{bmatrix} SO_1(n,1)/SO(n) & \text{(I)} \\ SU(n,1)/U(n) & \text{(II)} \\ SP(n,1)/(SP(n)\otimes SP(1)) & \text{(III)} \\ F_{4(-20)}/\,\mathrm{Spin}(9) & \text{(IV)} \end{bmatrix} \tag{A.3}$$

where $n \geqslant 2$. Then (for details see [15])

$$\begin{aligned} SO(p,q) &\stackrel{\text{def}}{=} \left\{ g \in GL(p+q,\mathbb{R}) \Big|^{g^t I_{pq} g = I_{pq}}_{\det g = 1} \right\} \\ {*SU}(p,q) &\stackrel{\text{def}}{=} \left\{ g \in GL(p+q,\mathbb{C}) \Big|^{g^t I_{pq} \bar{g} = I_{pq}}_{\det g = 1} \right\} \\ {*SP}(p,q) &\stackrel{\text{def}}{=} \left\{ g \in GL(2(p+q),\mathbb{C}) \Big|^{g^t J_{p+q} g = J_{p+q}}_{g^t K_{pq} \bar{g} = K_{pq}} \right\} \end{aligned} \tag{A.4}$$

where I_n is the identity matrix of order n and

$$I_{pq} = \begin{pmatrix} -I_p & 0 \\ 0 & I_q \end{pmatrix} \qquad J_n = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix} \qquad K_{pq} = \begin{pmatrix} I_{pq} & 0 \\ 0 & I_{pq} \end{pmatrix}. \tag{A.5}$$

The groups $SU(p,q)$ and $SP(p,q)$ are connected; the group $SO_1(p,q)$ is defined as the connected component of the identity in $SO(p,q)$, while $F_{4(-20)}$ is the unique real form of F_4 (with Dynkin diagram $\circ - \circ = \circ - \circ$) for which the character $(\dim X - \dim K)$ assumes the value (-20) [18]. We assume that if $G = SO(m,1)$ or $SU(q,1)$ then m is even and q is odd.

Let the data (G, K, Γ) be as in section 2, therefore G being one of the four groups of equation (A.3). The trace formula holds [22, 23]

$$\omega_\Gamma(t;b,\chi) = V \int_{\mathbb{R}} \mathrm{d}r\, \mathrm{e}^{-(r^2+b+\rho_0^2)t} |C(r)|^{-2} + \theta_\Gamma(t;b,\chi) \tag{A.6}$$

where by definition,

$$V \stackrel{\text{def}}{=} \frac{1}{4\pi} \chi(1)\ \mathrm{vol}(\Gamma\backslash G) \tag{A.7}$$

where χ is a finite-dimensional unitary representation (or a character) of Γ, and the number ρ_0 is associated with the positive restricted (real) roots of G (with multiplicity) with respect to a nilpotent factor N of G in an Iwasawa decomposition $G = KAN$. One has $\rho_0 = (n-1)/2, n, 2n+1, 11$ in cases (I)–(IV), respectively, in equation (A.3). Finally, the function $\theta_\Gamma(t;b,\chi)$ is defined as follows:

$$\theta_\Gamma(t;b,\chi) \stackrel{\text{def}}{=} \frac{1}{\sqrt{4\pi t}} \sum_{\gamma \in C_\Gamma - \{1\}} \chi(\gamma)\, t_\gamma\, j(\gamma)^{-1}\, C(\gamma)\, \mathrm{e}^{-(tb + t\rho_0^2 + t_\gamma^2/(4t))} \tag{A.8}$$

for a function $C(\gamma)$, $\gamma \in \Gamma$, defined on $\Gamma - \{1\}$ by

$$C(\gamma) \stackrel{\text{def}}{=} \mathrm{e}^{-\rho_0 t_\gamma} \Big| \det_{n_0} \left(\mathrm{Ad}\left(m_\gamma \mathrm{e}^{t_\gamma H_0}\right)^{-1} - 1 \right) \Big|^{-1}. \tag{A.9}$$

The notation used in equations (A.8) and (A.9) is the following. Let a_0, n_0 denote the Lie algebras of A, N. Since the rank of G is 1, $\dim a_0 = 1$ by definition, say $a_0 = \mathbb{R}H_0$ for a suitable basis vector H_0. One can normalize the choice of H_0 by $\sigma(H_0) = 1$, where $\sigma : a_0 \mapsto \mathbb{R}$ is the positive root which defines $n_0 = g_\sigma \oplus g_{2\sigma}$; for more detail see [15].

Since Γ is torsion free, each $\gamma \in \Gamma - \{1\}$ can be represented uniquely as some power of a primitive element $\delta: \gamma = \delta^{j(\gamma)}$ where $j(\gamma) \geqslant 1$ is an integer and δ cannot be written as γ_1^j for $\gamma_1 \in \Gamma$, $j > 1$ an integer. Taking $\gamma \in \Gamma$, $\gamma \neq 1$, one can find $t_\gamma > 0$ and $m_\gamma \in K$ satisfying $m_\gamma a = a m_\gamma$ for every $a \in A$ such that γ is G conjugate to $m_\gamma \exp(t_\gamma H_0)$, namely for some $g \in G$, $g\gamma g^{-1} = m_\gamma \exp(t_\gamma H_0)$. For Ad denoting the adjoint representation of G on its complexified Lie algebra, one can compute t_γ as follows [23]:

$$\mathrm{e}^{t_\gamma} = \max\{|c| \,|\, c = \text{an eigenvalue of } \mathrm{Ad}(\gamma)\} \tag{A.10}$$

in the case of $G = SO_1(m,1)$, with $|c|$ replaced by $|c|^{1/2}$ in the other cases of equation (A.3).

Appendix B. The spectral zeta function

The zeta function $\zeta_\Gamma(s|\mathcal{L})$ converges absolutely for $\mathrm{Re}\, s > d/2$, is holomorphic in s in this domain, and for $\mathrm{Re}\, s > d/2$

$$\zeta_\Gamma(s|\mathcal{L}) = \frac{\chi(1)\ \mathrm{Vol}(\Gamma\backslash G)}{4\pi}\mathcal{I}\big(s;\alpha^2\big) + \frac{1}{\Gamma(s)}T_\Gamma(s;\alpha,\chi) \tag{B.1}$$

where $\alpha^2 = b + \rho_0^2$ and [15]

$$\mathcal{I}\big(s;\alpha^2\big) = \int_{\mathbb{R}} \frac{|C(r)|^{-2}\,\mathrm{d}r}{\big(r^2+\alpha^2\big)^s} \tag{B.2}$$

$$\begin{aligned} T_\Gamma(s;\alpha,\chi) &= \frac{\pi^{-1/2}}{(2\alpha)^{s-\frac{1}{2}}}\sum_{\gamma\in C_\Gamma-\{1\}}\chi(\gamma)\,j(\gamma)^{-1}C(\gamma)t_\gamma^{s+\frac{1}{2}}K_{-s+\frac{1}{2}}(t_\gamma\alpha) \\ &= \frac{1}{\Gamma(1-s)}\int_0^\infty \Psi_\Gamma(t+\rho_0+\alpha;\chi)\big(2\alpha t+t^2\big)^{-s}\,\mathrm{d}t. \end{aligned} \tag{B.3}$$

The function $\Psi_\Gamma(s;\chi)$ is defined in [24]

$$\Psi_\Gamma(s;\chi) = \sum_{\gamma\in C_\Gamma-\{1\}}\chi(\gamma)\,t_\gamma\ j(\gamma)^{-1}\,C(\gamma)\,\mathrm{e}^{-(s-\rho_0)t_\gamma} \tag{B.4}$$

for $\mathrm{Re}\, s > 2\rho_0$. Thus Ψ_Γ is a holomorphic function in the $\frac{1}{2}$ plane $\mathrm{Re}\, s > 2\rho_0$ and admits a meromorphic continuation to the full complex plane. It can be shown that $\Psi_\Gamma(s;\chi) = Z'_\Gamma(s;\chi)/Z_\Gamma(s;\chi)$, where $Z_\Gamma(s;\chi)$ is a meromorphic suitable normalized Selberg zeta function attached to (G, K, Γ, χ) (see [3, 24–30]).

The suitable Harish–Chandra–Plancherel measure is given as follows:

$$|C(r)|^{-2} = \begin{bmatrix} C_G\pi r P(r)\tanh(\pi r), & \text{for } G = SO_1(2n,1), \\ C_G\pi r P(r)\tanh(\pi r/2), & \text{for } G = SU(q,1),\ q \text{ odd}, \\ & \text{or } G = SP(m,1),\ F_{4(-20)}, \\ C_G\pi r P(r)\coth(\pi r/2), & \text{for } G = SU(m,1),\ m \text{ even}, \\ C_G\pi P(r), & \text{for } G = SO_1(2n+1,1), \end{bmatrix} \tag{B.5}$$

while C_G is some constant depending on G, and where the $P(r)$ are even polynomials (with suitable coefficients a_{2l}) of degree $d-2$ for $G \neq SO(2n+1,1)$, and of degree $d-1 = 2n$ for $G = SO_1(2n+1,1)$ [3, 15].

For $\mathrm{Re}\, s > d/2$ and for $G \neq SO_1(m,1), SU(p,1)$ with m odd and p even we have [15]

$$\mathcal{I}\big(s;\alpha^2\big) = \tfrac{1}{2}\pi a(G)C_G W\big(s;\alpha^2,a(G)\big) \tag{B.6}$$

where

$$W\big(s;\alpha^2,a(G)\big) = \sum_{j=0}^{d/2-1} a_{2j}\,j!\sum_{l=0}^{j}\frac{\mathcal{K}_{j-l}\big(s-l-1;\alpha^2,a(G)\big)}{(j-l)!(s-1)(s-2)\ldots(s-(l+1))}. \tag{B.7}$$

For $G = SU(p,1)$ with p even and $\mathrm{Re}\, s > d/2 = p$,

$$\mathcal{I}\big(s;\alpha^2\big) = C_G\pi\left[\frac{\pi}{4}W\big(s;\alpha^2,\tfrac{1}{2}\pi\big) + \sum_{j=0}^{p-1} a_{2j}\mathcal{J}_j\big(s;\alpha^2,\tfrac{1}{2}\pi\big)\right]. \tag{B.8}$$

Finally, for $G = SO_1(2n+1,1)$ and $\mathrm{Re}\, s > (d/2) = (2n+1)/2$,

$$\begin{aligned}\mathcal{I}\big(s;\alpha^2\big) &= 2C_G\pi\sum_{j=0}^{n} a_{2j}\int_0^\infty \frac{r^{2j}\,\mathrm{d}r}{\big(r^2+\alpha^2\big)^s} \\ &= \frac{C_G\pi}{\Gamma(s)}\sum_{j=0}^{n} a_{2j}\alpha^{2(j+\frac{1}{2}-s)}\Gamma\big(j+\tfrac{1}{2}\big)\Gamma\big(s-j-\tfrac{1}{2}\big).\end{aligned} \tag{B.9}$$

In equations (B.7) and (B.8) the entire functions $\mathcal{K}_n(s;\delta,a)$ and $\mathcal{J}_n(s;\delta,a)$ are defined for $\delta, a > 0$ by

$$\mathcal{K}_n(s;\delta,a) = \int_{\mathbb{R}} \frac{r^{2n}\,\mathrm{sech}^2(ar)\,\mathrm{d}r}{\big(r^2+\delta^2\big)^s} \tag{B.10}$$

$$\mathcal{J}_n(s;\delta,a) = \int_{\mathbb{R}} \frac{r^{2n+1}\cosh(ar)\ \mathrm{sech}(ar)\,\mathrm{d}r}{\big(r^2+\delta^2\big)^s}. \tag{B.11}$$

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